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An Extension of Maximum and Anti-Maximum Principles to a Schrödinger Equation in $\mathbb{R}^2$

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Strong maximum and anti-maximum principles are extended to weak $L^2(\mathbb{R}^2)$-solutions u of the Schrödinger equation $-\Delta u + q(x)u - \lambda u = f(x)$ in $L^2(\mathbb{R}^2)$ in the following form: Let φ_1 denote the positive eigenfunction associated with the principal eigenvalue λ_1 of the Schrödinger operator $\mathcal{A} = -\Delta + q(x)\bullet$ in $L^2(\mathbb{R}^2)$. Assume that $q(x) \equiv q(|x|)$, f is a "sufficiently smooth" perturbation of a radially symmetric function, $f \not\equiv 0$ and $0 \leqslant f/\varphi_1 \leqslant C \equiv \text{const}$ a.e. in $\mathbb{R}^2$. Then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (-\infty, \lambda_1 + \delta)$ with $\lambda \neq \lambda_1$, the inequality $(\lambda_1 - \lambda)u \geqslant c\varphi_1$ holds a.e. in $\mathbb{R}^2$, where c is a positive constant depending upon f and λ. It is shown that such an inequality is valid if and only if the potential $q(x)$, which is assumed to be strictly positive and locally bounded, has a superquadratic growth as $|x| \to \infty$. This result is applied to linear and nonlinear elliptic boundary value problems in strongly ordered Banach spaces whose positive cone is generated by the eigenfunction φ_1. In particular, problems of existence and uniqueness are addressed. © 1999 Academic Press

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1. INTRODUCTION

Consider a linear partial differential equation with the Schrödinger operator,

$$-\Delta u + q(x)u - \lambda u = f(x) \qquad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where f is a given function satisfying $0 \leqslant f \not\equiv 0$ in $\mathbb{R}^N$ ($N \geqslant 1$), and λ stands for the spectral parameter. As usual, the Schrödinger operator takes the form $\mathscr{A} = -\Delta + q(x) \bullet$ in $L^2(\mathbb{R}^N)$ where Δ and $q(x) \bullet$, respectively, denote the selfadjoint Laplace operator and the pointwise multiplication operator by the potential q in $L^2(\mathbb{R}^N)$. Let φ_1 denote the positive eigenfunction of $\mathscr{A}$ associated with the principal eigenvalue λ_1. In quantum mechanics, φ_1 and λ_1 are called the ground state and the ground state energy, respectively.

DEFINITION 1.1. A function $u \in L^2(\mathbb{R}^N)$ is called φ_1-*positive* if there exists a constant $c > 0$ such that

$$u \geqslant c\varphi_1 \quad \text{almost everywhere in } \mathbb{R}^N. \tag{2}$$

Analogously, $u \in L^2(\mathbb{R}^N)$ is called φ_1-*negative* if there exists a constant $c > 0$ such that

$$u \leqslant -c\varphi_1 \quad \text{almost everywhere in } \mathbb{R}^N. \tag{3}$$

The φ_1-positivity (or φ_1-negativity) of a sufficiently smooth solution u to the equation (1.1), for $\lambda < \lambda_1$ (or $\lambda_1 < \lambda < \lambda_1 + \delta$, respectively), is an important result with numerous applications to both, linear and nonlinear elliptic problems in $\mathbb{R}^N$, see Alziary and Takáč [1]. Here, δ is a positive number depending upon f. Related boundary value problems in a bounded domain $\Omega \subset \mathbb{R}^N$ were studied in Amann [2], Clément and Peletier [4], Deimling [5], Fleckinger et al. [7, 8], Sweers [17], and Takáč [20]. Results of this type can be used for functional analytic treatment of the Cauchy problem (1) in the strongly ordered Banach space

$$X = \{u \in L^2(\mathbb{R}^N) : u/\varphi_1 \in L^\infty(\mathbb{R}^N)\} \tag{4}$$

endowed with the ordered norm

$$\|u\|_X = \inf\{C \in \mathbb{R} : |u| \leqslant C\varphi_1 \text{ almost everywhere in } \mathbb{R}^N\}. \tag{5}$$

The ordering "$\leqslant$" on X is the natural pointwise ordering of functions. This means that X is an ordered Banach space whose positive cone X_+ has nonempty interior $\mathring{X}_+$.

In a bounded domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\partial\Omega$, the φ_1-positivity of a sufficiently smooth solution u to an elliptic boundary value problem of second order $\mathscr{A}_\Omega u - \lambda u = f$, for $\lambda < \lambda_1$, follows from the strong maximum and boundary point principles, which are due to Bony [3] for weak solutions (see also P.-L. Lions [14]). The operator $\mathscr{A}_\Omega$ is assumed to satisfy standard boundary conditions, i.e., of Dirichlet, Neumann or Robin type. More precisely, assume that $f \in L^p(\Omega)$ for some $p > N$, and $0 \leqslant f \not\equiv 0$ in Ω. Let $u \in W^{2,p}(\Omega)$ be the weak solution of the boundary value problem $\mathscr{A}_\Omega u - \lambda u = f$ in $L^p(\Omega)$. Then there exists a constant $c > 0$ (depending upon f and λ) such that u satisfies the inequality (2) in Ω. In Protter and Weinberger [15, Chapt. 2, Theorem 10, p. 73], a similar result is referred to as the *generalized maximum principle*. It is complemented by the negativity of u, for $\lambda_1 < \lambda < \lambda_1 + \delta$, called an *anti-maximum principle* which is due to Clément and Peletier [4, Theorem 1, p. 222]. Since Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, their anti-maximum principle states equivalently that there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality (3) for the solution u of (1) is valid in Ω with a constant $c > 0$ (depending upon f and λ). This is the φ_1-negativity of u.

Let $u \in L^2(\mathbb{R}^N)$ be a sufficiently smooth solution of (1). The validity of (2) in $\mathbb{R}^N$, for $N \geqslant 1$, is established in Alziary and Takáč [1, Theorem 2.1, p. 284]. In our present work we prove the validity of (3) for the Schrödinger operator $\mathscr{A}$ in $L^2(\mathbb{R}^2)$. For the sake of convenience, in order to avoid complicated technicalities such as Sobolev imbeddings, we consider only the space dimension $N = 2$ and a radially symmetric potential $q(x) \equiv q(|x|)$. For either of (2) or (3) to be valid, it is necessary and sufficient that the potential $q(|x|)$, which is assumed to be strictly positive and locally bounded, have a superquadratic growth as $|x| \to \infty$. In particular, $q(|x|)$ must grow faster than $|x|^2$ as $|x| \to \infty$; the growth like $|x|^{2+\varepsilon}$ with any constant $\varepsilon > 0$ is sufficient (see Theorem 2.1 below). Thus, both (2) and (3) are in general false for the harmonic oscillator, i.e., for $q(x) = |x|^2$ in $\mathbb{R}^2$ (see Examples 4.1 and 4.2). We assume that $f(x)$ is a "sufficiently smooth" perturbation of a radially symmetric function, $f(x) = f_1(|x|) + f_2(x)$ for $x \in \mathbb{R}^2$.

Our methods combine the spectral theory for the selfadjoint linear operator $\mathscr{A} = -\Delta + q(x)\,\bullet$ in $L^2(\mathbb{R}^2)$, taking the spectral parameter λ near λ_1, with comparison results for radially symmetric solutions from M. Hoffmann-Ostenhof *et al.* [10, Lemma 3.2, p. 348] and Titchmarsh [21, Sect. 8.2, p. 165]. By spectral theory, for λ near λ_1, we can decompose the resolvent of $\mathscr{A}$ as

$$(\lambda I - \mathscr{A})^{-1} = (\lambda - \lambda_1)^{-1} \mathscr{P}_1 + \mathscr{H}(\lambda) \qquad \text{for} \quad 0 < |\lambda - \lambda_1| < \eta, \tag{6}$$

see, e.g., Sweers [17, Theorem 3.2(ii), p. 259] or Takáč [20, Eq. (6), p. 67]. Here, $\lambda \in \mathbb{C}$, $\eta > 0$ is small enough, $\mathscr{P}_1$ denotes the spectral projection onto the eigenspace spanned by φ_1, and $\mathscr{H}(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a holomorphic family of compact linear operators parametrized by λ with $|\lambda - \lambda_1| < \eta$. The main idea (and novelty) of our approach is to show that each of the linear operators $\{\mathscr{H}(\lambda) : |\lambda - \lambda_1| < \eta\}$ is bounded not only in $L^2(\mathbb{R}^2)$ but also in a suitable linear subspace Y (with a norm $\|\cdot\|_Y$) which is continuously and densely imbedded into the Banach space X defined in (4) with the norm (5). Clearly, given the Neumann series expansion of $\mathscr{H}(\lambda)$ for $|\lambda - \lambda_1| < \eta$, it suffices to show that the restriction $\mathscr{H}(\lambda_1)|_Y$ of $\mathscr{H}(\lambda_1)$ to Y is a bounded linear operator in Y. Indeed, we will show the boundedness of $\mathscr{H}(\lambda_1)|_{X^{1,2}}$ in a Banach space $X^{1,2}$ that contains only such perturbations of radially symmetric functions from X which are "sufficiently smooth" with respect to the angular variable. This is the main technical topic of our present article. Notice that the restriction $\mathscr{A}|_X$ of $\mathscr{A}$ to X is a closed, densely defined linear operator in X; the same statement is valid also for $\mathscr{A}|_{X^{1,2}}$.

In various common versions of the anti-maximum principle in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, besides the assumption $0 \leqslant f \not\equiv 0$ in Ω, it is only assumed that $f \in L^p(\Omega)$ for some $p > N$ (cf. Clément and Peletier [4, Theorem 1, p. 222], Sweers [17] or Takáč [20]). In our Remark 2.1 below we will discuss why for the Schrödinger operator in $L^2(\mathbb{R}^2)$ we need to assume more, namely, $f \in X$.

Standard applications of the inequalities (2) and (3) for the solution of (1) include the semilinear Cauchy problem

$$-\Delta u + q(x)u - \lambda u = f(x, u(x)) \qquad \text{in} \quad L^2(\mathbb{R}^2). \tag{SLS}$$

We will refer to this problem as the "semilinear Schrödinger equation", or the SLS equation, for brevity. Under certain reasonable hypotheses on the nonlinearity $f(x, u)$, we will be able to rewrite this equation as a fixed point problem. Consequently, a number of results from Amann [2], Deimling [5], Krasnosel'skiĭ and Zabreĭko [12], Takáč [18, 19], and many other publications exploring fixed points of strongly monotone or anti-monotone mappings may be applied to the SLS equation.

This article is organized as follows. In Section 2 we state our main result, Theorem 2.1. There, the inequalities (2) for $\lambda < \lambda_1$ and (3) for $\lambda_1 < \lambda < \lambda_1 + \delta$ are stated for the solution u of (1) under sufficient conditions on q and f. The necessity of $f \in X$ is suggested by Remark 2.1. In Section 3 we first state a few preliminary lemmas (partly taken from the literature) and then give the proof of the main result. Finally, in Section 4 we present an application and two examples. The semilinear Cauchy problem (SLS) can be rewritten as a fixed point equation to which Theorem 2.1 applies (Proposition 4.1). Examples 4.1 and 4.2, respectively, contain the harmonic oscillator for which the conclusions (16) and (17) of Theorem 2.1 are false.

2. THE MAIN RESULT

Notation. We denote by $\mathbb{R}^2$ the two-dimensional Euclidean space endowed with the inner product $x \cdot y$ and the norm $|x| = (x \cdot x)^{1/2}$, for $x, y \in \mathbb{R}^2$. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = (\mathbb{R}_+)^2 \subset \mathbb{R}^2$. For a set $M \subset \mathbb{R}^2$, we denote by ∂M ($\bar{M}$ and $\mathring{M}$, respectively) the boundary (closure and interior) of the set M in $\mathbb{R}^2$. We use analogous notation for sets in all Banach spaces.

Given a set $\Omega \subset \mathbb{R}^2$ and $1 \leqslant p \leqslant \infty$, we use the following standard Banach spaces of functions $f: \Omega \to \mathbb{R}$ (or $\mathbb{C}$), see e.g. Gilbarg and Trudinger [9, Chapt. 7]:

$L^p(\Omega)$, where Ω is Lebesgue measurable, is the Lebesgue space of all (equivalence classes of) Lebesgue measurable functions $f: \Omega \to \mathbb{R}$ with the norm

$$\|f\|_p \equiv \|f\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \begin{cases} (\int_\Omega |f(x)|^p \, dx)^{1/p} < \infty & \text{if} \quad 1 \leqslant p < \infty; \\ \operatorname{ess\,sup}_{x \in \Omega} |f(x)| < \infty & \text{if} \quad p = \infty. \end{cases}$$

$W^{k,p}(\Omega)$, where $k \geqslant 1$ is an integer and Ω open in $\mathbb{R}^2$, is the Sobolev space of all functions $f \in L^p(\Omega)$ whose all partial derivatives of order $\leqslant k$ also belong to $L^p(\Omega)$. The norm $\|f\|_{k,p} \equiv \|f\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$ is defined in a natural way.

The local Lebesgue and Sobolev spaces $L^p_{\text{loc}}(\Omega)$ and $W^{k,p}_{\text{loc}}(\Omega)$ are defined analogously. Finally, for Ω open in $\mathbb{R}^2$, $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$ is the space of all infinitely many times differentiable functions $f: \Omega \to \mathbb{R}$ with compact support. It is well-known that $\mathscr{D}(\mathbb{R}^2)$ is a dense linear subspace of both $L^p(\mathbb{R}^2)$ and $W^{k,p}(\mathbb{R}^2)$ for $1 \leqslant p < \infty$.

We denote by (r, θ) the polar coordinates in $\mathbb{R}^2$. Thus, for $x \in \mathbb{R}^2$ we write $x = (r\cos\theta, r\sin\theta)$ with $0 \leqslant r < \infty$ and $-\pi < \theta \leqslant \pi$. For a function $u: \mathbb{R}^2 \to \mathbb{R}$, we identify $u(x) \equiv u(r, \theta)$ if no confusion can arise. Given a 2π-periodic Lebesgue measurable function $f: \mathbb{R} \to \mathbb{R}$, we use the following abbreviations for the Lebesgue integral and the Lebesgue and Sobolev norms over the interval $[-\pi, \pi]$,

$$\oint f \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta; \qquad \|f\|_p^\circ \equiv \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta \right)^{1/p}; \tag{7}$$
$$\|f\|_{k,p}^\circ \equiv (1/2\pi)^{1/p} \, \|f\|_{W^{k,p}(-\pi,\pi)} \qquad \text{for} \quad 1 \leqslant p < \infty.$$

Next we introduce the Banach space $X^{1,2}$ of all Lebesgue measurable functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ having the following properties:

$$\frac{\partial f}{\partial \theta}(r, \bullet) \in L^2(-\pi, \pi) \qquad \text{for all} \quad r \geqslant 0$$

and there is a constant $C \geqslant 0$ such that

$$|f(r, \theta)| + \left(\oint \left| \frac{\partial f}{\partial \theta}(r, \vartheta) \right|^2 d\vartheta \right)^{1/2} \leqslant C\varphi_1(r)$$
$$\text{for almost every } r \geqslant 0 \text{ and } \theta \in [-\pi, \pi]. \tag{8}$$

The smallest such constant C defines the norm $\|f\|_{X^{1,2}}$ in $X^{1,2}$. Notice that, for $f(x) \equiv f(|x|)$, we have $f \in X^{1,2} \Leftrightarrow f \in X$ together with the norms $\|f\|_{X^{1,2}} = \|f\|_X$. Throughout this article, we refer to the functions from $X^{1,2}$ loosely as "sufficiently smooth" perturbations of radially symmetric functions.

In order to formulate our hypothesis on the potential $q(x)$, $x \in \mathbb{R}^2$, we first introduce the following class of auxiliary functions $Q(r)$ of $r \equiv |x|$, $R_0 \leqslant r < \infty$, for some $R_0 > 0$:

$$\begin{cases} Q(r) > 0, & Q \text{ is locally absolutely continuous,} \\ Q'(r) \geqslant 0, & \text{and} \quad \int_{R_0}^{\infty} Q(r)^{-1/2}\, dr < \infty. \end{cases} \tag{9}$$

We assume that the potential q is radially symmetric,

$$q(x) \equiv q(|x|), \qquad x \in \mathbb{R}^2, \tag{10}$$

where $q(r)$ is a Lebesgue measurable function satisfying the following hypothesis, with some auxiliary function $Q(r)$ obeying (9):

Hypothesis (**H**). *The potential* $q : \mathbb{R}_+ \to \mathbb{R}$ *is locally essentially bounded,* $q(r) \geqslant \text{const} > 0$ *for* $r \geqslant 0$, *and there exists a constant* $c_1 > 0$ *such that*

$$c_1 Q(r) \leqslant q(r) - \frac{1}{4r^2} \qquad \textit{for} \quad R_0 \leqslant r < \infty. \tag{11}$$

Notice that the fraction $1/4r^2$ in the inequality (11) is not essential and has been added for convenience in later applications; it can be left out.

Next we introduce the quadratic form

$$(v, w)_q \stackrel{\text{def}}{=} = \int_{\mathbb{R}^2} (\nabla v \cdot \nabla w + q(x)\, vw)\, dx \tag{12}$$

defined for every pair

$$v, w \in V_q \stackrel{\text{def}}{=} = \{ f \in L^2(\mathbb{R}^2) : (f, f)_q < \infty \}. \tag{13}$$

Notice that V_q is a Hilbert space with the inner product $(v, w)_q$ and the norm $\|v\|_{V_q} = ((v, v)_q)^{1/2}$. The set $\mathscr{D}(\mathbb{R}^2)$ is a dense linear subspace of V_q. By the Lax-Milgram theorem, the Schrödinger operator

$$\mathscr{A} = -\Delta + q(x) \bullet \qquad \text{in} \quad L^2(\mathbb{R}^2) \tag{14}$$

is defined to be the selfadjoint operator in $L^2(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} (\mathscr{A} v) w \, dx = (v, w)_q \qquad \text{for all} \quad v, w \in \mathscr{D}(\mathbb{R}^2). \tag{15}$$

We denote by $\mathscr{D}(\mathscr{A})$ its domain. The Banach space $\mathscr{D}(\mathscr{A})$ endowed with the graph norm is compactly embedded into $L^2(\mathbb{R}^2)$, by Rellich's theorem combined with $q(x) \to \infty$ as $|x| \to \infty$.

It is well-known that the principal eigenvalue λ_1 of $\mathscr{A}$ is given by

$$\lambda_1 = \inf \{ (f, f)_q : f \in V_q \text{ with } \|f\|_{L^2(\mathbb{R}^2)} = 1 \}, \qquad \lambda_1 > 0.$$

The eigenvalue λ_1 is simple with the eigenspace spanned by an eigenfunction $\varphi_1 \in \mathscr{D}(\mathscr{A})$ satisfying $\varphi_1 > 0$ throughout $\mathbb{R}^2$. We normalize φ_1 by the condition $\|\varphi_1\|_{L^2(\mathbb{R}^2)} = 1$. Since $q(x) \equiv q(|x|)$ for $x \in \mathbb{R}^2$, we must have also $\varphi_1(x) \equiv \varphi_1(|x|)$ for $x \in \mathbb{R}^2$. Furthermore, if $u \in \mathscr{D}(\mathscr{A})$ and $\mathscr{A} u = f \in L^2(\mathbb{R}^2)$ with $f \in L^p_{\text{loc}}(\mathbb{R}^2)$ for some p with $2 \leqslant p < \infty$, then the local L^p-regularity theory yields $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$, see Gilbarg and Trudinger [9, Theorem 9.15, p. 241]. In particular, if $p > 2$ then $u \in C^1(\mathbb{R}^2)$, by the Sobolev imbedding theorem [9, Theorem 7.10, p. 155]. It follows that also $\varphi_1 \in C^1(\mathbb{R}^2)$.

The following theorem about φ_1-positivity and φ_1-negativity of u is our main result:

THEOREM 2.1. *Let the hypothesis* (**H**) *be satisfied. Assume that* $u \in \mathscr{D}(\mathscr{A})$, $\mathscr{A} u - \lambda u = f \in L^2(\mathbb{R}^2)$, $\lambda \in \mathbb{R}$, *and* $f \geqslant 0$ *a.e. in* $\mathbb{R}^2$ *with* $f > 0$ *in some set of positive Lebesgue measure. Then, for every* $\lambda \in (-\infty, \lambda_1)$, *there exists a constant* $c > 0$ (*depending upon* f *and* λ) *such that*

$$u \geqslant c \varphi_1 \qquad \text{in} \quad \mathbb{R}^2. \tag{16}$$

Moreover, if also $f \in X^{1,2}$, *then there exists a positive number* δ (*depending upon* f) *such that, for every* $\lambda \in (\lambda_1, \lambda_1 + \delta)$, *the inequality*

$$u \leqslant -c \varphi_1 \qquad \text{in} \quad \mathbb{R}^2 \tag{17}$$

is valid with a constant $c > 0$ (*depending upon* f *and* λ).

Inequalities (16) and (17) are called the φ_1-*maximum* and φ_1-*anti-maximum principles*, respectively. We give a proof of Theorem 2.10 in Section 3. This proof makes use of the Laurent series (6). The first part of this theorem, Inequality (16), is proved in Alziary and Takáč [1, Theorem 2.1, p. 284] for more general potentials $q(x)$ which are assumed to be only "nearly" radially symmetric.

Theorem 2.1 is somewhat related also to a result in M. Hoffmann-Ostenhof [11, Theorem 1.4(i), p. 67]. However, the latter result deals with u which may change sign. On the other hand, our hypothesis (**H**) does not require a condition in [11, Eq. (1.7), p. 67], where the following restriction is imposed on $q(r)$:

$$\liminf_{r\to\infty} \frac{V'(r)}{V(r)}\, r \geqslant -1 \qquad \text{for} \quad V(r) = q(r) - \frac{1}{4r^2}.$$

As a special choice of the auxiliary function $Q(r)$ which obeys (9), we may take $Q(r) = r^{2+\varepsilon}$ with any constant $\varepsilon > 0$.

We can partly justify the necessity of the assumption $f \in X^{1,2}$ rather than only $f \in L^2(\mathbb{R}^2) \cap L^p_{\mathrm{loc}}(\mathbb{R}^2)$, for some $p > 2$, by the following remark and lemma.

Remark 2.1. Let the potential $q : \mathbb{R}^N \to \mathbb{R}$ $(N \geqslant 1)$ be Lebesgue measurable, nonnegative, and locally essentially bounded, with $q(x) \to +\infty$ as $|x| \to +\infty$. Let Y be an ordered Banach space with the following three properties: (i) Y is continuously imbedded into $L^2(\mathbb{R}^N)$; (ii) Y is a lattice endowed with the ordering from $L^2(\mathbb{R}^N)$; and (iii) Y is invariant under $\mathscr{H}(\lambda_1)$, i.e., $\mathscr{H}(\lambda_1) Y \subset Y$. Assume that both the weak maximum and anti-maximum principles for Eq. (1.1) are valid in the following form: if $0 \leqslant f \in Y$ and $u_\lambda = (\mathscr{A} - \lambda I)^{-1} f$ for $\lambda \in \mathbb{R}$ near λ_1 with $\lambda \neq \lambda_1$, then there exists a number $\delta \equiv \delta(f) > 0$ (depending upon f) such that

$$0 < |\lambda - \lambda_1| < \delta \Rightarrow (\lambda - \lambda_1) u_\lambda \leqslant 0.$$

This inequality is equivalent to

$$\begin{cases} \mathscr{H}(\lambda) f \leqslant (\lambda_1 - \lambda)^{-1}\, \mathscr{P}_1 f & \text{a.e. in } \mathbb{R}^N \quad \text{if} \quad 0 < \lambda_1 - \lambda < \delta(f); \\ -\mathscr{H}(\lambda) f \leqslant (\lambda - \lambda_1)^{-1}\, \mathscr{P}_1 f & \text{a.e. in } \mathbb{R}^N \quad \text{if} \quad 0 < \lambda - \lambda_1 < \delta(f), \end{cases} \tag{18}$$

where $\mathscr{H}(\lambda)$ is the pseudo-resolvent defined in Eq. (6) for $0 < |\lambda - \lambda_1| < \eta$.

Imposing a hypothesis slightly stronger than (18), and assuming (i), (ii) and (iii) above as well, we are able to prove the following result:

LEMMA 2.2. *Let us assume that the two-sided estimates in* (2.12) *hold on both sides of* λ_1, *i.e.*,

$$|\mathscr{H}(\lambda) f| \leqslant |\lambda - \lambda_1|^{-1} \mathscr{P}_1 f \text{ a.e. in } \mathbb{R}^N \qquad \text{for} \quad 0 < |\lambda - \lambda_1| < \delta(f).$$

Then $\mathscr{H}(\lambda_1)$ *restricts to a bounded linear operator from* Y *to* X *(see* (4) *and* (5)).

A proof of this lemma is given below.

We may view the boundedness of $\mathscr{H}(\lambda_1)|_Y : Y \to X$ as a condition which should be satisfied if one wants to show the validity of both, the weak maximum and anti-maximum principles. This condition indicates that it is reasonable to work exclusively with those functions f in Theorem 2.1 which satisfy $\mathscr{H}(\lambda_1) f \in X$. With $N = 2$, the space $Y = X^{1,2}$ defined above is a convenient choice for $f \in Y$; then we can combine standard L^2-methods with the Sobolev imbedding $W^{1,2}(-\pi, \pi) \hookrightarrow C[-\pi, \pi]$ for the angular variable. We do not know if $\mathscr{H}(\lambda_1)$ restricts to a bounded linear operator from X into itself, i.e., if we may take $Y = X$.

Proof of Lemma 2.2. Fix any $f \in Y \subset L^2(\mathbb{R}^N)$ with $f \geqslant 0$ in $\mathbb{R}^N$. As $\mathscr{P}_1$ is the orthogonal projection onto the eigenspace spanned by φ_1, with $\|\varphi_1\|_{L^2(\mathbb{R}^N)} = 1$, our assumption reads

$$|\mathscr{H}(\lambda) f| \leqslant |\lambda - \lambda_1|^{-1} \mathscr{P}_1 f = |\lambda - \lambda_1|^{-1} \left(\int_{\mathbb{R}^N} f \varphi_1 \, dx \right) \varphi_1$$

$$\text{a.e. in } \mathbb{R}^N \quad \text{for} \quad 0 < |\lambda - \lambda_1| < \delta(f). \tag{19}$$

We define $\delta(0) = \eta$. Now, Y being a lattice, write any $g \in Y \subset L^2(\mathbb{R}^N)$ as $g = g^+ - g^-$ where $g^+ \equiv \max\{g, 0\}$ and $g^- \equiv \max\{-g, 0\}$. It follows from (2.13) with f replaced by $g^\pm$ that

$$|\mathscr{H}(\lambda) g| \leqslant |\mathscr{H}(\lambda) g^+| + |\mathscr{H}(\lambda) g^-| \leqslant |\lambda - \lambda_1|^{-1} \left(\int_{\mathbb{R}^N} |g| \, \varphi_1 \, dx \right) \varphi_1$$

$$\text{a.e. in } \mathbb{R}^N \quad \text{for} \quad 0 < |\lambda - \lambda_1| < \delta(g), \tag{20}$$

where $\delta(g) = \min\{\delta(g^+), \delta(g^-)\}$. Moreover, the pseudo-resolvent $\{\mathscr{H}(\lambda) : |\lambda - \lambda_1| < \delta(g)\}$ satisfies the resolvent identity

$$\mathscr{H}(\lambda_1) - \mathscr{H}(\lambda) = (\lambda - \lambda_1)\, \mathscr{H}(\lambda)\, \mathscr{H}(\lambda_1). \tag{21}$$

Denoting $g_1 = \mathscr{H}(\lambda_1)\, g \in Y$, we use the estimate (20) also for g_1 in place of g. We apply (20) to Eq. (21) to obtain

$$|\mathscr{H}(\lambda_1)\, g|/\varphi_1 \leqslant |\lambda - \lambda_1|^{-1} \int_{\mathbb{R}^N} |g|\, \varphi_1\, dx + \int_{\mathbb{R}^N} |\mathscr{H}(\lambda_1)\, g|\, \varphi_1\, dx$$

$$\text{a.e. in } \mathbb{R}^N \text{ for } \quad 0 < |\lambda - \lambda_1| < \delta_1(g),$$

where $\delta_1(g) = \min\{\delta(g), \delta(g_1)\}$. Next we apply the Cauchy-Schwarz inequality to the integrals above and use $\|\varphi_1\|_{L^2(\mathbb{R}^N)} = 1$ to deduce

$$|\mathscr{H}(\lambda_1)\, g|/\varphi_1 \leqslant \delta_1(g)^{-1} \|g\|_{L^2(\mathbb{R}^N)} + \|\mathscr{H}(\lambda_1)\, g\|_{L^2(\mathbb{R}^N)} \quad \text{a.e. in } \mathbb{R}^N. \tag{22}$$

It follows from (22) that the linear operator $\mathscr{H}(\lambda_1)$ must map not only $L^2(\mathbb{R}^2)$ into itself, but also Y into X. Since $\mathscr{H}(\lambda_1)$ has closed graph in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, so does its restriction $\mathscr{H}(\lambda_1)|_Y$ to Y in $Y \times X$. Therefore, $\mathscr{H}(\lambda_1)|_Y \colon Y \to X$ is bounded by the closed graph theorem (see Yosida [22, Chapt. II, Sect. 6]). ∎

3. PROOF OF THE MAIN RESULT

We divide this section into the part with some preliminary lemmas and the part with the actual proof of our theorem.

3.1. *Preliminary Lemmas*

We start with a comparison result in the space dimension $N = 1$ proved in M. Hoffmann-Ostenhof et al. [10, Lemma 3.2, p. 348].

LEMMA 3.1. *Let $U_1(r)$ and $U_2(r)$ be two potentials satisfying U_1, $U_2 \in L^1_{\mathrm{loc}}(R, \infty)$ and $0 < \mathrm{const} \leqslant U_1 \leqslant U_2$ for $r \geqslant R$, where $0 < R < \infty$. Let all $f_1, f_2, f_1', f_2' \in L^2(R, \infty)$ be locally absolutely continuous in $[R, \infty)$, and let also $f_1 > 0$ and $f_2 > 0$ for $r \geqslant R$. Finally, assume that*

$$-f_1'' + U_1 f_1 \geqslant 0 \quad \text{and} \quad -f_2'' + U_2 f_2 \leqslant 0 \qquad \textit{for almost every} \quad r > R.$$

Then we have

$$\frac{f_1}{f_2} \geqslant \frac{f_1(R)}{f_2(R)} \quad \text{and} \quad \frac{f_1'}{f_1} \geqslant \frac{f_2'}{f_2} \qquad \textit{for every} \quad r \geqslant R.$$

Next we need a tight upper bound on the ratio f_1'/f_1 from Lemma 3.1. Instead of applying [11, Lemma 2.2, p. 68], we take advantage of a better upper bound taken from Titchmarsh [21, Sect. 8.2, p. 165].

LEMMA 3.2. *Let $U(r)$ be a nondecreasing function of $r \geqslant R$, with $U(R) > 0$, where $0 < R < \infty$. Let f and f' be locally absolutely continuous in $[R, \infty)$, with $f > 0$ for $r \geqslant R$ and $f(r) \to 0$ as $r \to \infty$. Assume that*

$$-f'' + Uf = 0 \qquad \textit{for almost every} \quad r > R.$$

Then we have $f'(r) \to 0$ as $r \to \infty$ and

$$-f'/f \geqslant U(r)^{1/2} \qquad \textit{for every} \quad r \geqslant R.$$

Now we combine Lemmas 3.1 and 3.1 in order to derive a similar upper bound for φ_1'/φ_1.

LEMMA 3.3. *Let the hypothesis* (**H**) *be satisfied. Then there exist positive constants γ_1 and R_1, with $R_1 \geqslant R_0$, such that*

$$\frac{1}{\varphi_1(r)^2 r} \int_r^\infty \varphi_1(\varrho)^2 \varrho \, d\varrho \leqslant \gamma_1 Q(r)^{-1/2} \qquad \textit{for every} \quad r \geqslant R_1. \tag{23}$$

Proof. We introduce the potentials

$$U(r) = \frac{1}{2} c_1 Q(r) \qquad \text{and} \qquad V(r) = q(r) - \lambda_1 - \frac{1}{4r^2} \qquad \text{for} \quad R_0 \leqslant r < \infty. \tag{24}$$

Hence, choosing a constant $R_1 \geqslant R_0$ large enough, we have $U(r) \leqslant V(r)$ for every $r \geqslant R_1$, by the hypothesis (0.0). Next we introduce the function $v(r) = r^{1/2} \varphi_1(r)$ and observe that

$$-v''(r) + V(r)\, v(r) = 0 \qquad \text{for a.e.} \quad r > R_1. \tag{25}$$

In addition, we have also $v, v' \in L^2(R_1, \infty)$ and $v > 0$ for $r \geqslant R_1$. All these properties of v follow from the equation $\mathscr{A}\varphi_1 - \lambda_1 \varphi_1 = 0$ for $\varphi_1 \in L^2(\mathbb{R}^2)$. We denote by u the weak solution in $W^{1,2}(R_1, \infty)$ of the boundary value problem

$$-u''(r) + U(r)\, u(r) = 0 \qquad \text{for a.e.} \quad r > R_1; \qquad u(R_1) = v(R_1). \tag{26}$$

Such a solution exists and is unique.

It follows from Lemma 3.1 that $u \geqslant v$ and $u'/u \geqslant v'/v$ for every $r \geqslant R_1$. Applying Lemma 3.2 to Eq. (26), we obtain $-u'/u \geqslant U(r)^{1/2}$ for every $r \geqslant R_1$. We combine these inequalities to arrive at $-v'/v \geqslant U(r)^{1/2}$ for every

$r \geqslant R_1$. Taking advantage of the last inequality, we now estimate the expression $v(r)^2$ from below, for $R_1 \leqslant r \leqslant R < \infty$:

$$v(r)^2 = v(R)^2 - 2 \int_r^R v(\varrho)\, v'(\varrho)\, d\varrho \geqslant 2 \int_r^R v(\varrho)^2\, U(\varrho)^{1/2}\, d\varrho.$$

Consequently, using (24) with Q nondecreasing, we obtain

$$v(r)^2 \geqslant 2U(r)^{1/2} \int_r^R v(\varrho)^2\, d\varrho = (2c_1 Q(r))^{1/2} \int_r^R v(\varrho)^2\, d\varrho.$$

Letting $R \to \infty$, we arrive at (23) with $\gamma_1 = (2c_1)^{-1/2}$. ▮

Finally, we show a comparison result for the selfadjoint operators $\mathscr{A}$ and $\mathscr{B}$ in $L^2(\mathbb{R}^2)$ defined by (15) and

$$\int_{\mathbb{R}^2} (\mathscr{B}v) w\, dx = (v, w)_q + \int_{\mathbb{R}^2} vw\, |x|^{-2}\, dx$$
$$\text{for all} \quad v, w \in \mathscr{D}(\mathbb{R}^2) \quad \text{with} \quad v(0) = w(0) = 0, \tag{27}$$

respectively. The reader is referred to Edmunds and Evans [6, Theorem VII.4.2, p. 384] or Reed and Simon [16, Theorem X.11, p. 161] for non-trivial details concerning the selfadjointness of $\mathscr{B}$. We recall that $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ denote the domains of $\mathscr{A}$ and $\mathscr{B}$. The symbols $\mathscr{A}^{1/2}$ and $\mathscr{B}^{1/2}$ stand for the positive-definite, selfadjoint square roots of $\mathscr{A}$ and $\mathscr{B}$ which are defined by spectral resolution (see Yosida [22, Chapt. XI, Sect. 12]).

LEMMA 3.4. *Let the hypothesis* (**H**) *be satisfied. If* $u \in \mathscr{D}(\mathscr{A}^{1/2})$ *is such that* $\int_{-\pi}^{\pi} u(r, \theta)\, d\theta = 0$ *for almost all* $r \geqslant 0$, *then the function* $x \mapsto \|u(r, \bullet)\|_2^\circ$ *satisfies* $\|u\|_2^\circ \in \mathscr{D}(\mathscr{B}^{1/2})$ *and*

$$\|\mathscr{B}^{1/2}\, \|u\|_2^\circ\|_{L^2(\mathbb{R}^2)} \leqslant \|\mathscr{A}^{1/2} u\|_{L^2(\mathbb{R}^2)} < \infty. \tag{28}$$

Proof. Let $u \in \mathscr{D}(\mathscr{A}^{1/2})$ and set $v(r) = \|u(r, \bullet)\|_2^\circ$ for $r \geqslant 0$. From the triangle inequality

$$|v(s) - v(r)| \leqslant \|u(s, \bullet) - u(r, \bullet)\|_2^\circ \qquad \text{for} \quad 0 \leqslant r < s < \infty$$

we easily derive

$$\left|\frac{dv}{dr}\right| \leqslant \left\|\frac{\partial u}{\partial r}\right\|^\circ \qquad \text{for almost all} \quad r \geqslant 0. \tag{29}$$

For every $f \in W^{1,2}(-\pi, \pi)$ such that $f(-\pi) = f(\pi)$, the Poincaré inequality reads

$$(\|f\|_2^\circ)^2 \leqslant \left| \oint f \, d\theta \right|^2 + (\|f'\|_2^\circ)^2.$$

Thus, assuming also $\int_{-\pi}^{\pi} u(r, \theta) \, d\theta = 0$ for almost all $r \geqslant 0$, we have

$$v \leqslant \left\| \frac{\partial u}{\partial \theta} \right\|_2^\circ \quad \text{for almost all } r \geqslant 0. \tag{30}$$

Consequently, we combine (15) and (27) to obtain

$$\begin{aligned}
&\|\mathscr{B}^{1/2} v\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad = \int_0^\infty \int_{-\pi}^{\pi} \left[\left(\frac{dv}{dr} \right)^2 + \left(\frac{1}{r^2} + q(r) \right) v^2 \right] d\theta \, r \, dr \\
&\quad \leqslant \int_0^\infty \int_{-\pi}^{\pi} \left[\left(\left\| \frac{\partial u}{\partial r} \right\|_2^\circ \right)^2 + \frac{1}{r^2} \left(\left\| \frac{\partial u}{\partial \theta} \right\|_2^\circ \right)^2 + q(r) v^2 \right] d\theta \, r \, dr \\
&\quad = \int_0^\infty \int_{-\pi}^{\pi} \left[\left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 + q(r) \, |u|^2 \right] d\theta \, r \, dr = \|\mathscr{A}^{1/2} u\|_{L^2(\mathbb{R}^2)}^2 < \infty.
\end{aligned}$$

It follows that $v \in \mathscr{D}(\mathscr{B}^{1/2})$ and (28) holds. ∎

Remark 3.1. For later use we notice that every function $u \in \mathscr{D}(\mathscr{B}^{1/2})$ satisfies $v \in C(\mathbb{R}^2)$ with $v(0) = 0$, where $v(r) = \|u(r, \bullet)\|_2^\circ$ for $r \geqslant 0$. To see this, we first observe that $v \in \mathscr{D}(\mathscr{B}^{1/2})$, by (29) combined with the radial symmetry of q. For $0 \leqslant \varrho' \leqslant \varrho < \infty$, we estimate the difference

$$v(\varrho)^2 - v(\varrho')^2 = 2 \int_{\varrho'}^{\varrho} v \, \frac{dv}{dr} \, dr = 2 \int_{\varrho'}^{\varrho} (r^{-1/2} v)(r^{1/2} v') \, dr$$

using the Cauchy-Schwarz inequality, thus arriving at

$$|v(\varrho)^2 - v(\varrho')^2| \leqslant 2 \left(\int_{\varrho'}^{\varrho} v^2 r^{-1} \, dr \right)^{1/2} \left(\int_{\varrho'}^{\varrho} |v'|^2 r \, dr \right)^{1/2}.$$

Since $v \in \mathscr{D}(\mathscr{B}^{1/2})$, we conclude that $v \in C(\mathbb{R}^2)$ and $v(0) = 0$ as well.

Remark 3.2. The principal eigenvalue $\tilde{\lambda}_1$ of $\mathscr{B}$ is given by

$$\tilde{\lambda}_1 = \inf \left\{ (f, f)_q + \int_{\mathbb{R}^2} |f|^2 \, |x|^{-2} \, dx : f \in \mathscr{D}(\mathscr{B}^{1/2}) \text{ with } \|f\|_{L^2(\mathbb{R}^2)} = 1 \right\}.$$

Similarly as for $\mathscr{A}$, the eigenvalue $\tilde{\lambda}_1$ is simple with the eigenspace spanned by an eigenfunction $\tilde{\varphi}_1 \in \mathscr{D}(\mathscr{B})$ satisfying $\tilde{\varphi}_1 > 0$ throughout $\mathbb{R}^2$. We normalize $\tilde{\varphi}_1$ by the condition $\|\tilde{\varphi}_1\|_{L^2(\mathbb{R}^2)} = 1$. Notice that the inequalities

$$0 < \lambda_1 = (\varphi_1, \varphi_1)_q \leqslant (\tilde{\varphi}_1, \tilde{\varphi}_1)_q < (\tilde{\varphi}_1, \tilde{\varphi}_1)_q + \int_{\mathbb{R}^2} \tilde{\varphi}_1^2 |x|^{-2}\, dx = \tilde{\lambda}_1$$

yield $0 < \lambda_1 < \tilde{\lambda}_1$. We conclude that the weak maximum principle for the linear partial differential equation

$$-\Delta u + (q(x) + |x|^{-2}) u - \lambda u = f(x) \qquad \text{in} \quad \mathbb{R}^2 \tag{3.9}$$

holds for every parameter value λ satisfying $\lambda < \tilde{\lambda}_1$. That is, if $\lambda < \tilde{\lambda}_1$, $u \in \mathscr{D}(\mathscr{B})$ and $\mathscr{B}u - \lambda u = f \geqslant 0$ in $\mathbb{R}^2$, then also $u \geqslant 0$ in $\mathbb{R}^2$. In particular, this statement holds for $\lambda = \lambda_1$.

3.2. *Proof of the Theorem*

We start our *proof of Theorem* 2.1 with the following orthogonal decomposition of the Hilbert space $L^2(\mathbb{R}^2)$ with respect to the standard inner product

$$(f, g) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} f(x)\, g(x)\, dx \qquad \text{for} \quad f, g \in L^2(\mathbb{R}^2).$$

We define its subspaces

$$H_1 = \{\alpha \varphi_1 : \alpha \in \mathbb{R}\} \subset L^2(\mathbb{R}^2); \tag{32}$$

$$H_2 = \{f \in L^2(\mathbb{R}^2) : f(x) \equiv f(|x|) \qquad \text{with} \quad \int_0^\infty f(r)\, \varphi_1(r) r\, dr = 0\}; \tag{33}$$

$$H_3 = \{f \in L^2(\mathbb{R}^2) : \int_{-\pi}^{\pi} f(r, \theta)\, d\theta = 0 \text{ for almost all } r \geqslant 0\}. \tag{34}$$

It is obvious that $L^2(\mathbb{R}^2) = H_1 \oplus H_2 \oplus H_3$ is an orthogonal decomposition. The corresponding orthogonal projections $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$, respectively, take the following forms, for each $f \in L^2(\mathbb{R}^2)$:

$$\mathscr{P}_1 f = \frac{(f, \varphi_1)}{(\varphi_1, \varphi_1)}\, \varphi_1; \tag{35}$$

$$\mathscr{P}_2 f = (I - \mathscr{P}_1) \oint f(\bullet, \theta)\, d\theta; \tag{36}$$

$$\mathscr{P}_3 f = \left(f - \oint f(\bullet, \theta)\, d\theta \right). \tag{37}$$

These projections commute with each other, by Fubini's theorem. Moreover, as $q(x) \equiv q(|x|)$, both $\mathscr{P}_1$ and the integral $\oint$ commute with the Schrödinger operator $\mathscr{A}$ defined by (2.9). Hence, all projections $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$ commute with $\mathscr{A}$.

We decompose also the Schrödinger operator $\mathscr{A}$ in $L^2(\mathbb{R}^2)$ accordingly: $\mathscr{A}_i = \mathscr{P}_i \mathscr{A}|_{H_i} : H_i \to H_i$; $i = 1, 2, 3$. By the Riesz-Schauder spectral theory (cf. Yosida [22, Chapt. X, Sect. 5]), we arrive at the following Laurent series for the resolvent of $\mathscr{A}$,

$$(\lambda I - \mathscr{A})^{-1} = (\lambda - \lambda_1)^{-1} \mathscr{P}_1 + (\lambda I - \mathscr{A}_2)^{-1} \mathscr{P}_2 + (\lambda I - \mathscr{A}_3)^{-1} \mathscr{P}_3$$

$$\text{for} \quad 0 < |\lambda - \lambda_1| < \eta. \tag{38}$$

Here, $\lambda \in \mathbb{C}$, $\eta > 0$ is small enough, and the resolvents $(\lambda I - \mathscr{A}_i)^{-1}$: $H_i \to H_i$; $i = 2, 3$, form a holomorphic family of compact linear operators parametrized by λ with $|\lambda - \lambda_1| < \eta$. Formula (38) was already used in Sweers [17, Theorem 3.2(ii), p. 259] and Takáč [20, Eq. (6), p. 67] to prove an anti-maximum principle of Hopf's type in a bounded domain with a smooth boundary.

In order to prove both, the φ_1-maximum and φ_1-anti-maximum principles, (2) and (3), respectively, we take advantage of the Laurent series (38). Clearly, it suffices to show that the resolvent $(\lambda I - \mathscr{A}_i)^{-1} : H_i \to H_i$ is a bounded linear operator from $\mathscr{P}_i X^{1,2}$ into X, for $i = 2, 3$ and $|\lambda - \lambda_1| < \eta$. Writing down the Neumann series for each resolvent $(\lambda I - \mathscr{A}_i)^{-1}$ about λ_1,

$$(\lambda I - \mathscr{A}_i)^{-1} = (\lambda_1 I - \mathscr{A}_i)^{-1} \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_1)^n (\lambda_1 I - \mathscr{A}_i)^{-n},$$

we find out that the boundedness of $(\lambda I - \mathscr{A}_i)^{-1} : \mathscr{P}_i X^{1,2} \to X$ follows from the boundedness of $(\lambda_1 I - \mathscr{A}_i)^{-1}$ from $\mathscr{P}_i X^{1,2}$ into itself, provided $|\lambda - \lambda_1| < \eta$ with $\eta > 0$ small enough. Furthermore, the number η depends exclusively upon the norms of $(\lambda_1 I - \mathscr{A}_i)^{-1}$ in $\mathscr{P}_i X^{1,2}$, for $i = 2, 3$. Recall that $\mathscr{P}_2 X^{1,2} = \mathscr{P}_2 X$. The boundedness of $(\lambda_1 I - \mathscr{A}_i)^{-1}$ from $\mathscr{P}_i X^{1,2}$ into itself follows from the next two propositions, for $i = 2, 3$, respectively:

PROPOSITION 3.5. *Let the hypothesis* (**H**) *be satisfied. Assume that* $u \in \mathscr{D}(\mathscr{A})$, $\mathscr{A}u - \lambda_1 u = f \in L^2(\mathbb{R}^2)$, *and f and u satisfy the following two properties*:

(i) $f(x) \equiv f(|x|)$, $(f, \varphi_1) = 0$ *and* $(u, \varphi_1) = 0$; *and*

(ii) $|f(r)|/\varphi_1(r) \leqslant C \equiv \text{const}$ *for a.e.* $r \geqslant 0$.

Then $u(x) \equiv u(|x|)$ *and there exists another constant* $\Gamma > 0$ (*depending exclusively upon the potential* q) *such that*

$$|u(r)|/\varphi_1(r) \leqslant \Gamma C \qquad \text{for all} \quad r \geqslant 0. \tag{39}$$

Equivalently, the operator norm of $(\lambda_1 I - \mathscr{A}_2)^{-1}$ *in* $\mathscr{P}_2 X$ *is* $\leqslant \Gamma$.

Proof. From $f \in H_2$ and $u = (\lambda_1 I - \mathscr{A}_2)^{-1} f$ we obtain $u(x) \equiv u(|x|)$. Standard regularity theory from Gilbarg and Trudinger [9] applied to the equation $\mathscr{A}u - \lambda_1 u = f \in L^2(\mathbb{R}^2)$ with $f \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ yields $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ whenever $2 \leqslant p < \infty$, and thus $u \in C^1(\mathbb{R}^2)$ for $p > 2$. So we may rewrite this equation for $f(r)$ and $u(r)$ in the form

$$-u''(r) - \frac{1}{r}\, u'(r) + (q(r) - \lambda_1)\, u(r) = f(r) \qquad \text{for a.e.} \quad r > 0. \tag{40}$$

Similarly, $\varphi_1(r)$ satisfies the equation

$$-\varphi_1''(r) - \frac{1}{r}\, \varphi_1'(r) + (q(r) - \lambda_1)\, \varphi_1(r) = 0 \qquad \text{for a.e.} \quad r > 0. \tag{41}$$

In addition, we have also $u'(0) = 0$ and $\varphi_1'(0) = 0$. Now we set $g = f/\varphi_1$, $w = u/\varphi_1$ and combine Eqs. (40) and (41), thus obtaining the following initial value problem for the unknown function $w'(r)$,

$$\begin{cases} -w''(r) - \dfrac{1}{r}\, w'(r) - 2[\ln \varphi_1(r)]'\, w'(r) = g(r) & \text{for a.e.} \quad r > 0; \\ -w'(0) = 0. \end{cases} \tag{42}$$

Using the variation-of-constants formula, we obtain

$$\begin{aligned} w'(r) &= -\frac{1}{\varphi_1(r)^2\, r} \int_0^r g(\varrho)\, \varphi_1(\varrho)^2\, \varrho \, d\varrho \\ &= \frac{1}{\varphi_1(r)^2\, r} \int_r^\infty g(\varrho)\, \varphi_1(\varrho)^2\, \varrho \, d\varrho \qquad \text{for all} \quad r > 0. \end{aligned} \tag{43}$$

The last equality follows from $(g\varphi_1, \varphi_1) = 0$. Consequently, we can integrate Eq. (43) as follows,

$$w(R) = w(0) + \int_0^R w'(r)\, dr \qquad \text{for all} \quad R \geqslant 0, \tag{44}$$

where the condition $(w\varphi_1, \varphi_1) = 0$ forces

$$w(0) = -\int_0^\infty \left(\int_0^R w'(r)\, dr \right) \varphi_1(R)^2\, R\, dR. \tag{45}$$

We finish our proof by showing that there exists a constant $\Gamma_1 > 0$ (depending exclusively upon the potential q) such that

$$\int_0^\infty |w'(r)|\, dr \leqslant \Gamma_1 C. \tag{46}$$

Let $R_1 \geqslant R_0$ be the constant from Lemma 3.3. Applying $|g(r)| \leqslant C$ for $0 \leqslant r \leqslant R_1$ (a.e.) to the first formula in Eq. (43), we conclude that

$$\begin{aligned} |w'(r)| &\leqslant \frac{C}{\varphi_1(r)^2 r} \int_0^r \varphi_1(\varrho)^2\, \varrho\, d\varrho \leqslant C\, \frac{M(R_1)^2}{r} \int_0^r \varrho\, d\varrho \\ &= \frac{M(R_1)^2 r}{2}\, C \qquad \text{for} \quad 0 \leqslant r \leqslant R_1, \end{aligned} \tag{47}$$

where

$$M(R_1) = \max_{0 \leqslant \varrho \leqslant r \leqslant R_1} \frac{\varphi_1(\varrho)}{\varphi_1(r)} < \infty.$$

Similarly, applying $|g(r)| \leqslant C$ for $R_1 \leqslant r < \infty$ (a.e.) to the second formula in Eq. (43), we obtain

$$|w'(r)| \leqslant \frac{C}{\varphi_1(r)^2 r} \int_r^\infty \varphi_1(\varrho)^2\, \varrho\, d\varrho \qquad \text{for} \quad R_1 \leqslant r < \infty. \tag{48}$$

We insert (23) into (48) to obtain

$$|w'(r)| \leqslant C\gamma_1 Q(r)^{-1/2} \qquad \text{for} \quad R_1 \leqslant r < \infty. \tag{49}$$

Finally, we integrate the estimates (47) and (49) rendering the desired bound (46), where we may choose the constant

$$\Gamma_1 = (M(R_1)\, R_1/2)^2 + \gamma_1 \int_{R_1}^\infty Q(r)^{-1/2}\, dr < \infty.$$

The estimate (39) follows from a combination of (44) and (45) with (46). ∎

PROPOSITION 3.6. *Let the hypothesis* (**H**) *be satisfied. Assume that* $u \in \mathscr{D}(\mathscr{A})$, $\mathscr{A}u - \lambda_1 u = f \in L^2(\mathbb{R}^2)$, *and* f *and* u *satisfy the following two properties*:

(i) $\oint f(r, \theta)\, d\theta = 0$ *and* $\oint u(r, \theta)\, d\theta = 0$ *for a.e.* $r > 0$; *and*

(ii) $f \in X^{1,2}$ (*see the inequality* (8)).

Then there exists a constant $\Gamma > 0$ (*depending exclusively upon the potential* q) *such that*

$$\|u\|_{X^{1,2}} \leqslant \Gamma \, \|f\|_{X^{1,2}}. \tag{50}$$

Equivalently, the operator norm of $(\lambda_1 I - \mathscr{A}_3)^{-1}$ *in* $\mathscr{P}_3 X^{1,2}$ *is* $\leqslant \Gamma$.

Proof. Recall that $\mathscr{A}_3 - \lambda_1 I$ has a bounded inverse in $H_3 = \mathscr{P}_3(L^2(\mathbb{R}^2))$. Given any $h \in \mathbb{R} \setminus \{0\}$, both functions

$$u^h(r, \theta) = h^{-1}[u(r, \theta + h) - u(r, \theta)] \qquad \text{and}$$

$$f^h(r, \theta) = h^{-1}[f(r, \theta + h) - f(r, \theta)]$$

of $(r, \theta) \in (0, \infty) \times (-\pi, \pi)$ belong to H_3 and satisfy also $u^h = (\mathscr{A}_3 - \lambda_1 I)^{-1} f^h$. Letting $h \to 0$ and recalling $f \in X^{1,2}$, we obtain

$$\frac{\partial u}{\partial \theta} = (\mathscr{A}_3 - \lambda_1 I)^{-1} \frac{\partial f}{\partial \theta} \qquad \text{in } H_3.$$

Equivalently, the functions $u_\theta \equiv \partial u / \partial \theta$ and $f_\theta \equiv \partial f / \partial \theta$ satisfy

$$u_\theta \in \mathscr{D}(\mathscr{A}) \cap H_3, \;\; f_\theta \in H_3, \qquad \text{and} \qquad (\mathscr{A} - \lambda_1 I)\, u_\theta = f_\theta. \tag{51}$$

Consequently, Lemma 3.4 yields $\|u_\theta\|_2^\circ \in \mathscr{D}(\mathscr{B}^{1/2})$.

Next we will show that the inequality

$$(\mathscr{B} - \lambda_1 I)\, \|u_\theta\|_2^\circ \leqslant \|f_\theta\|_2^\circ \qquad \text{in } \mathbb{R}^2 \tag{52}$$

is valid in the following weak sense,

$$(\|u_\theta\|_2^\circ, w)_q + \int_{\mathbb{R}^2} (r^{-2} - \lambda_1)\, \|u_\theta\|_2^\circ\, w\, dx \leqslant \int_{\mathbb{R}^2} \|f_\theta\|_2^\circ\, w\, dx$$

$$\text{for all} \quad w \in \mathscr{D}(\mathscr{B}^{1/2}) \quad \text{with} \quad w \geqslant 0 \quad \text{in} \quad \mathbb{R}^2. \tag{53}$$

To this end we introduce the set Ψ of all test functions $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that ψ is Lipschitz continuous with compact support and $\psi(0) = 0$. It is clear that Ψ is an ordered linear space with the positive cone $\Psi_+ = \{\psi \in \Psi : \psi \geqslant 0 \text{ in } \mathbb{R}_+\}$. The positive and negative parts of ψ, defined by $\psi^+ \equiv \max\{\psi, 0\}$ and $\psi^- \equiv \max\{-\psi, 0\}$, respectively, also satisfy $\psi^+, \psi^- \in \Psi$. Hence, Ψ is a vector lattice.

Given any $\varepsilon > 0$ and $\psi \in \Psi_+$, we define the function

$$w_\varepsilon(r, \theta) = u_\theta(r, \theta)\, \psi(r) \left(\varepsilon^2 + \oint |u_\theta(r, \theta')|^2 \, d\theta' \right)^{-1/2}. \tag{54}$$

It follows that

$$\frac{\partial w_\varepsilon}{\partial r} = \left(\frac{\partial u_\theta}{\partial r}\, \psi + u_\theta \frac{d\psi}{dr} \right) (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}$$
$$- u_\theta \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-3/2} \oint u_\theta \frac{\partial u_\theta}{\partial r} \, d\theta' \tag{55}$$

and

$$\frac{\partial w_\varepsilon}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta}\, \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}. \tag{56}$$

As a consequence of $u_\theta \in \mathscr{D}(\mathscr{A}^{1/2})$ we obtain $w_\varepsilon \in \mathscr{D}(\mathscr{B}^{1/2})$, using the Cauchy-Schwarz inequality. Next, we first take the inner product in $L^2(\mathbb{R}^2)$ of Eq. (51) with the function w_ε and then use Eq. (15) to obtain

$$\begin{aligned}
&((\mathscr{A} - \lambda_1 I)\, u_\theta, w_\varepsilon)_{L^2(\mathbb{R}^2)} \\
&\quad = (u_\theta, w_\varepsilon)_q - \lambda_1 (u_\theta, w_\varepsilon)_{L^2(\mathbb{R}^2)} \\
&\quad = \int_0^\infty \int_{-\pi}^{\pi} \left(\left| \frac{\partial u_\theta}{\partial r} \right|^2 \psi + u_\theta \frac{\partial u_\theta}{\partial r} \frac{d\psi}{dr} \right) (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2} \, d\theta \, r \, dr \\
&\qquad - \int_0^\infty \int_{-\pi}^{\pi} u_\theta \frac{\partial u_\theta}{\partial r}\, \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-3/2} \left(\oint u_\theta \frac{\partial u_\theta}{\partial r} \, d\theta' \right) d\theta \, r \, dr \\
&\qquad + \int_0^\infty \int_{-\pi}^{\pi} \left(\frac{1}{r^2} \left| \frac{\partial u_\theta}{\partial \theta} \right|^2 + (q(r) - \lambda_1)\, |u_\theta|^2 \right) \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2} \, d\theta \, r \, dr \\
&\quad = (f_\theta, w_\varepsilon)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Hence, employing the identity

$$\|u_\theta\|_2^\circ \frac{d}{dr} \|u_\theta\|_2^\circ = \frac{1}{2} \frac{d}{dr} (\|u_\theta\|_2^\circ)^2 \frac{1}{2} \frac{d}{dr} \oint |u_\theta(r, \theta)|^2 \, d\theta = \oint u_\theta \frac{\partial u_\theta}{\partial r} \, d\theta,$$

we conclude that

$$\begin{aligned}
&2\pi \int_0^\infty \left[\left(\left\| \frac{\partial u_\theta}{\partial r} \right\|_2^\circ \right)^2 \psi + \|u_\theta\|_2^\circ \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right) \frac{d\psi}{dr} \right] (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}\, r\, dr \\
&\qquad - 2\pi \int_0^\infty (\|u_\theta\|_2^\circ)^2 \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right)^2 \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-3/2}\, r\, dr \\
&\qquad + 2\pi \int_0^\infty \left[\frac{1}{r^2} \left(\left\| \frac{\partial u_\theta}{\partial \theta} \right\|_2^\circ \right)^2 + (q(r) - \lambda_1)(\|u_\theta\|_2^\circ)^2 \right] \\
&\qquad \times \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}\, r\, dr \\
&\quad = (f_\theta, w_\varepsilon)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Recall that $\psi \in \Psi_+$ is arbitrary. In order to estimate the left-hand side of this equation from below, we apply the triangle and Poincaré inequalities (29) and (30) as follows,

$$\begin{aligned}
&2\pi \int_0^\infty \left[\left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right)^2 \psi + \|u_\theta\|_2^\circ \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right) \frac{d\psi}{dr} \right] (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}\, r\, dr \\
&\qquad - 2\pi \int_0^\infty (\|u_\theta\|_2^\circ)^2 \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right)^2 \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-3/2}\, r\, dr \\
&\qquad + 2\pi \int_0^\infty \left[\frac{1}{r^2} (\|u_\theta\|_2^\circ)^2 + (q(r) - \lambda_1)(\|u_\theta\|_2^\circ)^2 \right] \\
&\qquad \times \psi (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2}\, r\, dr \\
&\quad \leqslant (f_\theta, w_\varepsilon)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Simplifying the left-hand side and applying the Cauchy-Schwarz inequality to the inner product on the right-hand side, we arrive at

$$\begin{aligned}
&\int_0^\infty \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right) \|u_\theta\|_2^\circ (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2} \frac{d\psi}{dr}\, r\, dr \\
&\qquad + \varepsilon^2 \int_0^\infty \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ \right)^2 (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-3/2} \psi r\, dr \\
&\qquad + \int_0^\infty \left(\frac{1}{r^2} + q(r) - \lambda_1 \right) (\|u_\theta\|_2^\circ)^2 (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2} \psi r\, dr \\
&\qquad \leqslant \int_0^\infty \|f_\theta\|_2^\circ \|u_\theta\|_2^\circ (\varepsilon^2 + (\|u_\theta\|_2^\circ)^2)^{-1/2} \psi r\, dr \leqslant \int_0^\infty \|f_\theta\|_2^\circ \psi r\, dr.
\end{aligned}$$

Finally, we let $\varepsilon \to 0+$ to obtain

$$((\mathscr{B} - \lambda_1 I)\, \|u_\theta\|_2^\circ,\ \psi)_{L^2(\mathbb{R}^2)}$$

$$= 2\pi \int_0^\infty \left(\frac{d}{dr}\, \|u_\theta\|_2^\circ\right) \frac{d\psi}{dr}\, r\, dr + 2\pi \int_0^\infty \left(\frac{1}{r^2} + q(r) - \lambda_1\right) \|u_\theta\|_2^\circ\, \psi r\, dr$$

$$\leqslant 2\pi \int_0^\infty \|f_\theta\|_2^\circ\, \psi r\, dr = (\|f_\theta\|_2^\circ,\ \psi)_{L^2(\mathbb{R}^2)} \qquad \text{for all} \quad \psi \in \Psi_+.$$

Since the linear space Ψ is dense in $\mathscr{D}(\mathscr{B}) \cap (H_1 \oplus H_2)$, the latter one being a Banach lattice endowed with the graph norm from $\mathscr{D}(\mathscr{B})$, the last inequality yields (53) as claimed.

Let us set $C = \|f\|_{X^{1,2}}$. Observe that $\|f_\theta\|_2^\circ \leqslant C\varphi_1(r)$ for every $r \geqslant 0$. Since $\lambda_1 < \tilde{\lambda}_1$, we can employ the weak maximum principle to conclude that Ineq. (53) forces $\|u_\theta\|_2^\circ \leqslant Cv_1(r)$ for every $r \geqslant 0$, where $v_1 \in \mathscr{D}(\mathscr{B}^{1/2})$ denotes the weak solution of

$$(\mathscr{B} - \lambda_1 I) v_1 = \varphi_1(r) \qquad \text{in} \quad \mathbb{R}^2, \tag{57}$$

that is, v_1 satisfies

$$(v_1,\ w)_q + \int_{\mathbb{R}^2} (r^{-2} - \lambda_1) v_1 w\, dx$$

$$= \int_{\mathbb{R}^2} \varphi_1 w\, dx \qquad \text{for all} \quad w \in \mathscr{D}(\mathscr{B}^{1/2}). \tag{58}$$

Now we fix an arbitrary number $\varrho \geqslant 0$. Taking ε arbitrary, $0 < \varepsilon < 1$, we make the special choice $w(r) = \varphi_1(r)\, \psi_\varepsilon(r)$ with a piecewise linear function

$$\psi_\varepsilon(r) \stackrel{\text{def}}{=} \min\{\varepsilon^{-1} \cdot \max\{r - \varrho,\ 0\},\ 1\} \qquad \text{for} \quad r \geqslant 0.$$

Clearly, we have $w \in \mathscr{D}(\mathscr{B}^{1/2})$, and Eq. (58) becomes

$$\int_0^\infty v_1'(\varphi_1 \psi_\varepsilon)'\, r\, dr + \int_0^\infty \left(\frac{1}{r^2} + q(r) - \lambda_1\right) v_1 \varphi_1 \psi_\varepsilon r\, dr = \int_0^\infty \varphi_1^2 \psi_\varepsilon r\, dr.$$

Making use of the equality

$$\int_{\mathbb{R}^2} [(\mathscr{A} - \lambda_1 I)\varphi_1]\, v_1 \psi_\varepsilon\, dx = 0,$$

we thus arrive at

$$\int_0^\infty (v_1'\varphi_1 - v_1\varphi_1')\,\psi_\varepsilon' r\,dr + \int_0^\infty v_1\varphi_1\psi_\varepsilon r^{-1}\,dr$$

$$= \varepsilon^{-1}\int_\varrho^{\varrho+\varepsilon} (v_1'\varphi_1 - v_1\varphi_1') r\,dr + \int_\varrho^\infty v_1\varphi_1\psi_\varepsilon r^{-1}\,dr = \int_\varrho^\infty \varphi_1^2\psi_\varepsilon r\,dr. \tag{59}$$

Standard regularity theory from Gilbarg and Trudinger [9] applied to the equation (57) with $\varphi_1 \in C^1(\mathbb{R}^2)$ yields $v_1 \in C^1(\mathbb{R}^2\backslash\{0\})$. Consequently, letting $\varepsilon \to 0+$ in Eq. (59), we arrive at

$$\varphi_1(\varrho)^2\,\varrho w_1'(\varrho) + \int_\varrho^\infty w_1(r)\,\varphi_1(r)^2\,r^{-1}\,dr = \int_\varrho^\infty \varphi_1(r)^2\,r\,dr \tag{60}$$

for every $\varrho > 0$, where $w_1 = v_1/\varphi_1$.

Let $R_1 > 0$ be the constant specified in the proof of Proposition 3.5 above. Since $v_1 \in \mathscr{D}(\mathscr{B}^{1/2})$, v_1 is continuous at $0 \in \mathbb{R}^2$ with $v_1(0) = 0$, by Remark 3.1. Hence, $w_1 \in C(\mathbb{R}_+)$ yields

$$M(R_1) = \max_{0 \leqslant r \leqslant R_1} w_1(r) < \infty. \tag{61}$$

To show that the function $w_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded, we now estimate its derivative in the interval $[R_1, \infty)$. First using (60) and then applying (23), we have

$$w_1'(\varrho) \leqslant \frac{1}{\varphi_1(\varrho)^2\,\varrho}\int_\varrho^\infty \varphi_1(r)^2\,r\,dr \leqslant \gamma_1 Q(\varrho)^{-1/2} \qquad \text{for} \quad \varrho \geqslant R_1. \tag{62}$$

We integrate this estimate over any compact interval $[R_1, R]$, for $R_1 \leqslant R < \infty$, thus arriving at

$$w_1(R) \leqslant w_1(R_1) + \gamma_1 \int_{R_1}^R Q(r)^{-1/2}\,dr$$

$$\leqslant \Gamma_1 \overset{\text{def}}{=} M(R_1) + \gamma_1 \int_{R_1}^\infty Q(r)^{-1/2}\,dr < \infty. \tag{63}$$

Finally, we combine the estimates (61) and (63) to deduce that w_1 must be bounded on $\mathbb{R}_+$ by the constant $\Gamma_1 > 0$. Thus,

$$\|v\|_2^\circ/\varphi_1(r) \leqslant C v_1(r)/\varphi_1(r) = C w_1(r) \leqslant C\Gamma_1 \qquad \text{for all} \quad r \geqslant 0.$$

The estimate (50) follows immediately from this result supplemented by f, $u \in H_3$ and the continuity of the Sobolev imbedding $W^{1,2}(-\pi, \pi) \hookrightarrow C[-\pi, \pi]$. ∎

Proof of Theorem 2.1. Here we finish the proof of our main result. The inequality (16) is proved in Alziary and Takáč [1, Theorem 2.1, p. 284]. Our present method is able to cover only the case $f \in X^{1,2}$ and $0 < |\lambda - \lambda_1| < \eta$ which we now prove. We apply Propositions 3.5 and 3.6 to the Laurent series (38). It follows that

$$(\lambda_1 - \lambda)(\mathscr{A} - \lambda I)^{-1} f = \mathscr{P}_1 f + (\lambda - \lambda_1)[(\lambda I - \mathscr{A}_2)^{-1} \mathscr{P}_2 f + (\lambda I - \mathscr{A}_3)^{-1} \mathscr{P}_3 f] \quad \text{for} \quad 0 < |\lambda - \lambda_1| < \eta. \tag{64}$$

Taking $\lambda \in \mathbb{R}$ and $\eta > 0$ small enough, it is clear that the right-hand side of Eq. (3.42) belongs to $\mathring{X}_+$, by $\mathscr{P}_1 f \in \mathring{X}_+$. More precisely, there exist positive constants c and δ (depending upon f) such that

$$(\lambda_1 - \lambda) u = (\lambda_1 - \lambda)(\mathscr{A} - \lambda I)^{-1} f \geqslant c\varphi_1 \quad \text{in} \quad \mathbb{R}^2, \quad \text{for every} \quad \lambda \in \mathbb{R} \quad \text{with} \quad 0 < |\lambda - \lambda_1| < \delta.$$

We have proved Theorem 2.1. ∎

4. APPLICATION AND EXAMPLES

We commence with a standard application of Theorem 2.1 to a semilinear Cauchy problem whose linear part is a Schrödinger operator and the nonlinear part is a subhomogeneous pointwise substitution operator. Let us consider the semilinear Cauchy problem

$$-\Delta u + q(x)u - \lambda u = f(x, u(x)) \quad \text{in} \quad L^2(\mathbb{R}^2). \tag{65}$$

We assume that the potential $q(x) \equiv q(|x|)$, $x \in \mathbb{R}^2$, satisfies the hypothesis (**H**), $\lambda \in \mathbb{R}$ is the spectral parameter, and the function $f(x, u)$ of $(x, u) \in \mathbb{R}^2 \times \mathbb{R}$ satisfies the following three hypotheses:

HYPOTHESES. (f1) $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., the function $f(\bullet, u)$ is Lebesgue measurable in $\mathbb{R}^2$ for every $u \in \mathbb{R}$, and the function $f(x, \bullet)$ is continuous in $\mathbb{R}$ for almost every $x \in \mathbb{R}^2$.

(f2) The function $f(x, u) \equiv f(r, \theta, u)$ is differentiable with respect to θ and u with both partial derivatives $(\partial f/\partial \theta)(r, \theta, u)$ and $(\partial f/\partial u)(r, \theta, u)$ being also Carathéodory functions, i.e., Lebesgue measurable in $(r, \theta) \in \mathbb{R}_+ \times [-\pi, \pi]$ for every $u \in \mathbb{R}$, and continuous in $u \in \mathbb{R}$ for almost all $(r, \theta) \in \mathbb{R}_+ \times [-\pi, \pi]$.

(f3) $f(\bullet, 0)/\varphi_1 \in L^\infty(\mathbb{R}^2)$ and there exists a positive constant M_0 such that

$$\left|\frac{\partial f}{\partial u}(r, \theta, u)\right| \leqslant M_0 \qquad \text{for almost all } (r, \theta) \in \mathbb{R}_+ \times [-\pi, \pi] \text{ and for all } u \in \mathbb{R}. \tag{66}$$

(f4) There exists a positive function $M_1 \in L^2(-\pi, \pi)$ such that

$$\left|\frac{\partial f}{\partial \theta}(r, \theta, u)\right| \leqslant M_1(\theta)(\varphi_1(r) + |u|)$$

$$\text{for almost all } (r, \theta) \in \mathbb{R}_+ \times [-\pi, \pi] \text{ and for all } u \in \mathbb{R}. \tag{67}$$

We set

$$m_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{(x, u) \in \mathbb{R}^2 \times \mathbb{R}} \frac{\partial f}{\partial u} \geqslant -M_0.$$

We study the SLS equation (65) in the ordered Banach space defined in (1.4) endowed with the ordered norm (4) for $N = 2$. Of course, the ordering "$\leqslant$" on X is defined by $u \leqslant v$ if and only if $u(x) \leqslant v(x)$ for a.e. $x \in \mathbb{R}^2$. The mapping $u \mapsto u/\varphi_1 : X \to L^\infty(\mathbb{R}^2)$ defines an isomorphism of the Banach lattices X onto $L^\infty(\mathbb{R}^2)$. The positive cone $X_+ = \{u \in X : u \geqslant 0 \text{ in } X\}$ of X has nonempty interior $\mathring{X}_+$. We use also the Banach space $X^{1,2}$ with the norm $\|\bullet\|_{X^{1,2}}$ defined by (8).

In what follows μ and λ are any fixed numbers with $m_0 + \mu > 0$ and either $\lambda - \mu < \lambda_1$ or else $\lambda_1 < \lambda - \mu < \lambda_1 + \delta$. Our main goal is to prove the following result for two solutions $u, v \in X^{1,2}$ of the SLS equation (65) satisfying $u \geqslant v$.

PROPOSITION 4.1. *Assume that K is a compact set in $X^{1,2}$ such that $\emptyset \neq K \subset X_+ \setminus \{0\}$. Then there exists a positive number δ (depending upon K) with the following property: if $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $u, v \in X^{1,2}$ satisfy Eq. (65) together with $u \geqslant v$ throughout $\mathbb{R}^2$, then the function*

$$x \ (\in \mathbb{R}^2) \mapsto f(x, u(x)) - f(x, v(x)) + \mu(u(x) - v(x)) \tag{68}$$

must belong to $X_+ \setminus K$.

To prove this result, we rewrite Eq. (65) as the fixed point problem for the mapping $\mathscr{F} : v \mapsto \mathscr{F}(v) = w$ defined by

$$-\Delta w + q(x) w + (\mu - \lambda) w = f(x, v(x)) + \mu v \qquad \text{in} \quad X. \tag{69}$$

We define the pointwise substitution operator $\mathcal{N}$ in X by

$$(\mathcal{N}v)(x) = f(x, v(x)) + \mu v(x) \qquad \text{for a.e.} \quad x \in \mathbb{R}^2 \text{ and every } v \in X.$$

Then $\mathcal{N}$ maps X into itself with $\mathcal{N}(X^{1,2}) \subset X^{1,2}$ as well, and for all $v_1, v_2 \in X$ with $v_1 \geqslant v_2$ we have

$$\mathcal{N}v_1 - \mathcal{N}v_2 \geqslant (m_0 + \mu)(v_1 - v_2) \geqslant 0 \qquad \text{in} \quad X. \tag{70}$$

In particular, if $v_1 - v_2$ belongs to X_+ ($X_+ \setminus \{0\}$ or $\mathring{X}_+$, respectively), then so does $\mathcal{N}v_1 - \mathcal{N}v_2$. Moreover, Hypothesis (f3) implies that $\mathcal{N} : X \to X$ is uniformly Lipschitz continuous, whereas Hypotheses (f2), (f3) and (f4) guarantee that the restriction $\mathcal{N}|_{X^{1,2}} : X^{1,2} \to X^{1,2}$ of $\mathcal{N}$ to $X^{1,2}$ is continuous.

Finally, we define the mapping $\mathcal{F} : X^{1,2} \to X^{1,2}$ by

$$\mathcal{F}v = (\mathcal{A} + (\mu - \lambda)I)^{-1}\mathcal{N}v \qquad \text{for every} \quad v \in X^{1,2}.$$

This mapping is defined in all of X for $\lambda - \mu < \lambda_1$. Given any $v \in X^{1,2}$, the weak solution $w \in L^2(\mathbb{R}^2)$ of Eq. (69) has the form $w = \mathcal{F}v$. Consequently, every weak solution $u \in X^{1,2}$ of the SLS equation (65) satisfies the fixed point equation $\mathcal{F}u = u$ in $X^{1,2}$. We deduce from Theorem 2.1 and (70) that $\mathcal{F}$ has the following property:

LEMMA 4.2. *The mapping* $\mathcal{F} : X^{1,2} \to X^{1,2}$ *is* strongly monotone (strongly anti-monotone, *respectively*) *in* X *if* $\lambda - \mu < \lambda_1$ (*if* $\lambda_1 < \lambda - \mu < \lambda_1 + \delta$), *i.e., for every pair* $v_1, v_2 \in X^{1,2}$ *we have* $v_2 - v_1 \in X_+ \setminus \{0\} \Rightarrow \mathcal{F}v_2 - \mathcal{F}v_1 \in \mathring{X}_+$ ($\mathcal{F}v_2 - \mathcal{F}v_1 \in -\mathring{X}_+$).

Since the case $\lambda - \mu < \lambda_1$ is treated extensively in Alziary and Takáč [1, Example 4.1, p. 291], we adapt our current setting to the case $\lambda_1 < \lambda - \mu < \lambda_1 + \delta$. We summarize the most important properties of the mapping $\mathcal{F} : X^{1,2} \to X^{1,2}$ in the following lemma:

LEMMA 4.3. (i) *Let* K *be a compact set in* $X^{1,2}$ *such that* $\varnothing \neq K \subset X_+ \setminus \{0\}$. *Then there exists a positive number* δ (*depending upon* K) *with the following property*: *if* $\lambda \in (\lambda_1, \lambda_1 + \delta)$ *and* $u \in \mathcal{D}(\mathcal{A})$ *satisfies* $\mathcal{A}u - \lambda u = f \in K$, *then the inequality*

$$u \leqslant -c\varphi_1 \qquad \text{in} \quad \mathbb{R}^2 \tag{71}$$

is valid with a constant $c > 0$ *depending upon* K *and* λ, *but not on* f.

(ii) *Let* μ *and* λ *be any fixed numbers with* $m_0+\mu>0$ *and* $\lambda_1<\lambda-\mu<\lambda_1+\delta$. *The mapping* $\mathscr{F}: X^{1,2}\to X^{1,2}$ *is continuous and for every pair* $v_1, v_2\in X^{1,2}$ *we have* (*cf.* (70))

$$\mathscr{N}v_2-\mathscr{N}v_1\in K \Rightarrow \mathscr{F}v_2-\mathscr{F}v_1\in -\mathring{X}_+.$$

(iii) *Let* μ *and* λ *be as in Part* (ii). *In addition, let* B *be a closed bounded set in* $X^{1,2}$. *Hence,* B *is bounded also in* X *and* $L^2(\mathbb{R}^2)$. *Then the restriction* $\mathscr{F}|_B$ *of* $\mathscr{F}$ *to* B *is continuous as a mapping from* $X^{1,2}$ *into itself and uniformly Lipschitz continuous as a mapping from* $L^2(\mathbb{R}^2)$ *into itself. Moreover, the set* $\mathscr{F}(B)$ *is bounded in* $X^{1,2}$ *and has compact closure in* $L^2(\mathbb{R}^2)$.

(iv) *Finally, for* $u\in X^{1,2}$, *Eq.* (65) *holds in* $L^2(\mathbb{R}^2)$ *if and only if* $\mathscr{F}u=u$.

Proof. (i) follows from Ineq. (17) and Propositions 3.5 and 3.6 combined with a finite covering argument for the set K.

(ii) is a consequence of Theorem 2.1.

(iii) follows from the hypotheses (f2) through (f4). The closure of $\mathscr{F}(B)$ in $L^2(\mathbb{R}^2)$ is compact by the compactness of $(\mathscr{A}+(\mu-\lambda)I)^{-1}$ in $L^2(\mathbb{R}^2)$.

(iv) follows from our definition of the mapping $\mathscr{F}$. ∎

For such mappings $\mathscr{F}$, fixed point problems have been studied extensively, for example, in Amann [2, Sect. 12], Deimling [5], and Krasnosel'skiĭ and Zabreĭko [12] to mention only a few publications.

Proof of Proposition 4.1. Let λ and $u, v\in X^{1,2}$ satisfy the hypotheses of Proposition 4.1. Notice that the function defined in (68) takes the form $\mathscr{N}u-\mathscr{N}v$. Moreover, $u\geqslant v$ implies $\mathscr{N}u\geqslant\mathscr{N}v$. To prove $\mathscr{N}u-\mathscr{N}v\in X_+\setminus K$, assume the contrary, that is, $\mathscr{N}u-\mathscr{N}v\in K$. By Lemma 4.3, Part (ii), this forces $\mathscr{F}u-\mathscr{F}v\in -\mathring{X}_+$. Since both u and v are fixed points of $\mathscr{F}$, we arrive at the contradiction $u(x)<v(x)$ for a.e. $x\in\mathbb{R}^2$. ∎

Our two examples feature the well-known harmonic oscillator, i.e., $q(x)=|x|^2$ for $x\in\mathbb{R}^2$. They show that the estimates (16) and (17) in Theorem 2.1, respectively, are false for this particular choice of the potential q. The following counterexample to the φ_1-maximum principle (2) is analyzed in Alziary and Takáč [1, Example 4.1, p. 291]. Here we only sketch the main ideas.

EXAMPLE 4.1. We consider the harmonic oscillator in $\mathbb{R}^N$, $N\geqslant 1$,

$$-\Delta u+|x|^2u=f(x) \qquad \text{in} \quad L^2(\mathbb{R}^N), \tag{72}$$

where $f \in L^\infty(\mathbb{R}^N)$ satisfies $0 \leqslant f \not\equiv 0$ and $f(x) = 0$ for $|x| \geqslant R$, with a positive constant R. Consequently, Eq. (72) possesses a unique C^1-solution u. It is shown in [1, Example 4.1, p. 291] that

$$0 < u(x) \leqslant Cr^{-\alpha}\varphi_1(r) \quad \text{for } r = |x| \geqslant R, \tag{73}$$

where α is an arbitrary number with $0 < \alpha < N/2$ and C is a positive constant. Thus, the conclusion (16) of Theorem 2.1 is violated for the potential $q(x) = |x|^2$.

Now we construct a counterexample to the anti-maximum principle (3) for $q(x) = |x|^2$.

Example 4.2. We consider a generalization of the harmonic oscillator in $\mathbb{R}^N$, $N \geqslant 1$,

$$-\Delta u + |x|^2\, u - \lambda u = f(x) \qquad \text{in} \quad L^2(\mathbb{R}^N), \tag{74}$$

where $f \in L^\infty(\mathbb{R}^N)$, $0 \leqslant f \not\equiv 0$, and $\lambda \in \mathbb{R}$ is the spectral parameter which takes values from a vicinity of λ_1 with $\lambda \neq \lambda_1$. Recall that λ_1 denotes the principal eigenvalue of the Schrödinger operator $\mathscr{A} = -\Delta + |x|^2 \bullet$ in $L^2(\mathbb{R}^N)$. Clearly, we have $\lambda_1 = N$ and $\varphi_1(x) = \exp(-\frac{1}{2}|x|^2)$ for $x \in \mathbb{R}^N$. Furthermore, the solution u of Eq. (74) is continuously differentiable in $\mathbb{R}^N$, by the arguments used in the proof of Theorem 2.1 above. From now on we assume that f is radially symmetric, that is, $f(x) \equiv f(|x|)$. It follows that also $u(x) \equiv u(|x|)$, by uniqueness. Define the functions

$$w(r) = u(r)/\varphi_1(r) \qquad \text{and} \qquad g(r) = f(r)/\varphi_1(r) \qquad \text{for} \quad r \geqslant 0.$$

Notice that Eq. (74) entails

$$\begin{cases} -w''(r) - \dfrac{N-1}{r}\, w' + 2rw' - (\lambda - \lambda_1)\, w = g(r) & \text{for a.e.} \quad r > 0; \\ w'(0) = 0, \end{cases} \tag{75}$$

which in turn implies for the derivatives $w_1 = dw/dr$ and $g_1 = dg/dr$ that

$$\begin{cases} -w_1''(r) - \dfrac{N-1}{r}\, w_1' + 2rw_1' + \dfrac{N-1}{r^2}\, w_1 - (\lambda - \lambda_1 - 2)\, w_1 \\ = g_1(r) \qquad \text{for a.e.} \quad r > 0; \qquad w_1(0) = 0. \end{cases} \tag{76}$$

Consequently, the weak maximum principle applies to Eq. (76) for the unknown function w_1 whenever $\lambda < \lambda_1 + 2$. Hence, by linearity, we can also compare solutions.

Indeed, assume that $\lambda < \lambda_1 + 2$ and g_1 is a nonnegative measurable function of $r \geqslant 0$ satisfying

$$\int_0^\infty g_1(r)^2\, \varphi_1^2(r)\, r^{N-1}\, dr < \infty.$$

In order to prove that $w_1 \geqslant 0$ for a.e. $r \geqslant 0$, suppose the contrary, i.e., the function $w_1^- \equiv \max\{-w_1,\ 0\}$ does not vanish almost everywhere in $\mathbb{R}_+$. We first multiply Eq. (76) by the factor $w_1^-(r)\, \varphi_1^2(r) r^{N-1}$, then integrate the result over the half-line $(0, \infty)$, and finally apply integration by parts, thus obtaining

$$\int_0^\infty w_1'(w_1^-)'\, \varphi_1^2 r^{N-1}\, dr + (N-1) \int_0^\infty (w_1/r)(w_1^-/r)\, \varphi_1^2 r^{N-1}\, dr$$
$$- (\lambda - \lambda_1 - 2) \int_0^\infty w_1 w_1^-\, \varphi_1^2 r^{N-1}\, dr = \int_0^\infty g_1 w_1^-\, \varphi_1^2 r^{N-1}\, dr.$$

It follows that

$$-\int_0^\infty ((w_1^-)')^2\, \varphi_1^2 r^{N-1}\, dr - (N-1) \int_0^\infty (w_1^-/r)^2\, \varphi_1^2 r^{N-1}\, dr$$
$$+ (\lambda - \lambda_1 - 2) \int_0^\infty (w_1^-)^2\, \varphi_1^2 r^{N-1}\, dr = \int_0^\infty g_1 w_1^-\, \varphi_1^2 r^{N-1}\, dr. \qquad (77)$$

Recaling $\lambda < \lambda_1 + 2$ and $g_1(r) \geqslant 0$ for a.e. $r \geqslant 0$, we deduce that all integrals on the left-hand side of Eq. (77) must vanish, i.e., $w_1^-(r) = 0$ for a.e. $r \geqslant 0$ as claimed.

Now set $\alpha = (\lambda - \lambda_1)/2$; in what follows it is assumed that $0 < \alpha < 1$. Define the function

$$\tilde{w}(r) = \frac{1}{2\alpha}\, ((1 + r^2)^{\alpha/2} - C_\alpha) \qquad \text{for} \quad r \geqslant 0. \qquad (78)$$

Here, $C_\alpha \in \mathbb{R}$ is a constant to be determined later. Notice that $\tilde{w}'(0) = 0$. We replace w by $\tilde{w}$ in Eq. (75), thus obtaining the corresponding function $\tilde{g}$ in place of g on the right-hand side of Eq. (75),

$$\tilde{g}(r) = C_\alpha - \tfrac{1}{2}(N + \alpha)(1 + r^2)^{(\alpha/2) - 1}$$
$$- \tfrac{1}{2}(2 - \alpha)(1 + r^2)^{(\alpha/2) - 2} \qquad \text{for} \quad r \geqslant 0. \qquad (79)$$

Consequently, Eq. (76) holds for $d\tilde{w}/dr$ and $d\tilde{g}/dr$ in place of w_1 and g_1, respectively, where

$$\frac{d\tilde{g}}{dr}(r) = \frac{1}{2} r(N+\alpha)(2-\alpha)(1+r^2)^{(\alpha/2)-2} + \frac{1}{2} r(2-\alpha)(4-\alpha)(1+r^2)^{(\alpha/2)-3} \quad \text{for} \quad r \geqslant 0. \tag{80}$$

Next we define the function $g(r)$ of $r \geqslant 0$ by

$$g(r) = (N+4)(1-(1+r^2)^{-1/2}) \quad \text{for} \quad r \geqslant 0. \tag{81}$$

Hence, $f = \varphi_1 g \in X$ is radially symmetric, $0 \leqslant g \leqslant N+4$ throughout $\mathbb{R}_+$, and

$$\frac{dg}{dr}(r) = (N+4)\, r(1+r^2)^{-3/2} \quad \text{for} \quad r \geqslant 0. \tag{82}$$

Taking into account that $0 < \alpha < 1$, we combine Eqs. (80) and (82) to arrive at

$$\frac{d\tilde{g}}{dr}(r) \leqslant \frac{1}{2} r[(N+\alpha)(2-\alpha) + (2-\alpha)(4-\alpha)](1+r^2)^{(\alpha/2)-2} \leqslant (N+4)\, r(1+r^2)^{(\alpha/2)-2} \leqslant \frac{dg}{dr}(r) \quad \text{for} \quad r \geqslant 0. \tag{83}$$

Thus, we have $d\tilde{g}/dr \leqslant dg/dr$ throughout $\mathbb{R}_+$.

Finally, let $u = \varphi_1 w$ be the solution of Eq. (74) in $L^2(\mathbb{R}^N)$ corresponding to $f = \varphi_1 g$. Notice that, since g depends only upon r, the same is true of w. As $d(g - \tilde{g})/dr \geqslant 0$ throughout $\mathbb{R}_+$, it is readily seen that also $d(w - \tilde{w})/dr \geqslant 0$ throughout $\mathbb{R}_+$, by the weak maximum principle applied to Eq. (76) for the unknown function $d(w - \tilde{w})/dr$. We conclude that

$$w(r) \geqslant \tilde{w}(r) = \frac{1}{\lambda - \lambda_1} ((1+r^2)^{(\lambda-\lambda_1)/4} - C_{(\lambda-\lambda_1)/2}) \quad \text{for} \quad r \geqslant 0, \tag{84}$$

where the constant $C_{(\lambda-\lambda_1)/2}$ is determined from the equation

$$w(0) = \frac{1}{\lambda - \lambda_1} (1 - C_{(\lambda-\lambda_1)/2}).$$

Consequently, from Ineq. (4.20) we obtain $w(r) \to +\infty$ as $r \to +\infty$. Therefore, for the function $f = \varphi_1 g$ constructed above, the anti-maximum principle (3) cannot be valid for any λ with $\lambda_1 < \lambda < \lambda_1 + 2$.

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